# CHAPTER 1

# COMPLEX STRESSES

## 1.0. Introduction

The previous studies were confined to direct stresses caused on planes at right angles to the line of action of applied loads and to shear stresses. In this chapter we advance our studies to the stresses caused on planes inclined to the line of action of applied loads. However complex the system of forces acting on a body be, yet it is possible to locate three mutually perpendicular planes which carry purely normal stresses. Planes within a material such that the resultant stresses across them are wholly normal stresses or planes across which no shearing stresses occur are termed as principal planes and the stresses acting on these planes are called principal stresses. The stress intensity on one of these planes at a point is more than on the other two planes and on the other one it is least of all. Below we proceed to investigate a few typical cases.

## 1.1. Stresses on oblique planes

Consider the general case, shown in Fig. 1.1, of a bar under direct load F giving rise to stress $\sigma_y$ vertically.

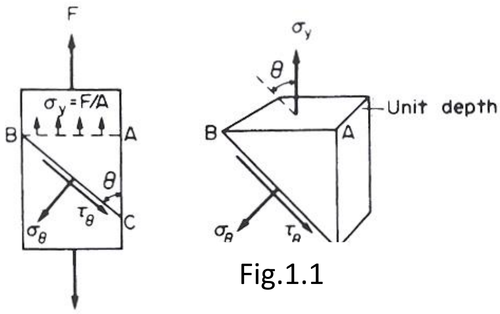


Fig.1.1

Let the block be of unit depth; then considering the equilibrium of forces on the triangular portion ABC resolve forces perpendicular to BC,

$$\sigma_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \sin\theta$$

But AB = BC sin θ

$$\therefore \quad \boldsymbol{\sigma_\theta = \sigma_y \sin^2\theta} \qquad ...(1)$$

Now resolving forces parallel to BC,

$$\tau_\theta \times BC \times 1 = \sigma_y \times AB \times 1 \times \cos\theta$$

Again AB = BC sin θ,

$$\therefore \quad \tau_\theta = \sigma_y \sin\theta\cos\theta$$

$$= \tfrac{1}{2}\boldsymbol{\sigma_y} \sin 2\boldsymbol{\theta} \qquad \text{...(2)}$$

The stresses on the inclined plane, therefore, are not simply the resolutions of $\sigma_y$ perpendicular and tangential to that plane. The direct stress $\sigma_\theta$ has a maximum value of $\sigma_y$ when $\theta = 90^0$ whilst the shear stress $\tau_\theta$ has a maximum value of $\frac{1}{2}\sigma_y$ when $\theta = 45°$.

Thus any material whose yield stress in shear is less than half that in tension or compression will yield initially in shear under the action of direct tensile or compressive forces.

This is evidenced by the typical "cup and cone" type failure in tension tests of ductile specimens such as low carbon steel where failure occurs initially on planes at 45° to the specimen axis. Similar effects occur in compression tests on, for example, timber where failure is again due to the development of critical shear stresses on 45° planes.

## 1.2. <u>Material subjected to pure shear</u>

Consider the element shown in Fig. 1.2 to which shear stresses have been applied to the sides AB and DC. Complementary shear stresses of equal value but of opposite effect are then set up on sides AD and BC in order to prevent rotation of the element. Since the applied and complementary shears are of equal value on the x and y planes, they are both given the symbol $\tau_{xy}$.

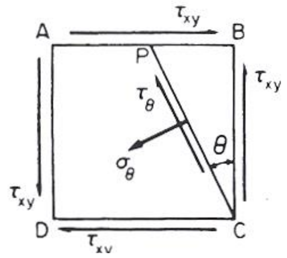


Fig. 1.2

Consider now the equilibrium of portion PBC. Resolving normal to PC assuming unit depth,

$$\sigma_\theta \times PC = \tau_{xy} \times BC\sin\theta + \tau_{xy} \times PB\cos\theta$$

$$= \tau_{xy} \times PC\cos\theta\sin\theta + \tau_{xy} \times PC\sin\theta\cos\theta$$

$$\therefore \quad \boldsymbol{\sigma_\theta = \tau_{xy} \sin 2\theta} \qquad \text{...(3)}$$

The maximum value of $\sigma_\theta$ is $\tau_{xy}$ when $\theta = 45°$. Similarly, resolving forces parallel to PC,

$$\tau_\theta \times PC = \tau_{xy} \times PB\sin\theta - \tau_{xy} BC\cos\theta$$

$$= \tau_{xy} \times PC\sin^2\theta - \tau_{xy} \times PC\cos^2\theta$$

$$\therefore \quad \boldsymbol{\tau_\theta = -\tau_{xy} \cos 2\theta} \qquad \text{...(4)}$$

The negative sign means that the sense of $\tau_\theta$ is opposite to that assumed in Fig. 1.2. The maximum value of $\tau_\theta$ is $\tau_{xy}$ when θ = 0° or 90° and it has a value of zero when θ = 45°, i.e. on the planes of maximum direct stress.

Further consideration of eqns. (3) and (4) shows that the system of pure shear stresses produces an equivalent direct stress system as shown in Fig. 1.3, one set compressive and one tensile, each at 45° to the original shear directions, and equal in magnitude to the applied shear.

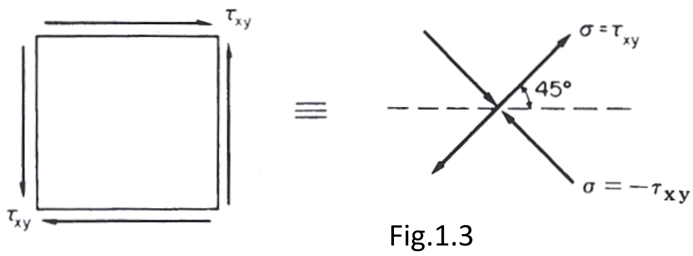



Fig.1.3

This has great significance in the measurement of shear stresses or torques on shafts using strain gauges where the gauges are arranged to record the direct strains at 45° to the shaft axis.

Practical evidence of the theory is also provided by the failure of brittle materials in shear. A shaft of a brittle material subjected to torsion will fail under direct stress on planes at 45° to the shaft axis. (This can be demonstrated easily by twisting a piece of blackboard chalk in one’s hands; Tearing of a wet cloth when it is being wrung out is also attributed to the direct stresses introduced by the applied torsion.

## 1.3. Material subjected to two mutually perpendicular direct stresses

Consider the rectangular element of unit depth shown in Fig. 1.4 subjected to a system of two direct stresses, both tensile, at right angles, $\sigma_x$ and $\sigma_y$. For equilibrium of the portion ABC, resolving perpendicular to AC,

$$\sigma_\theta \times AC \times 1 = \sigma_x \times BC \times 1 \times \cos\theta + \sigma_y \times AB \times 1 \times \sin\theta$$

$$= \sigma_x \times AC \cos^2\theta + \sigma_y \times AC \sin^2\theta$$

∴ $$\sigma_\theta = \tfrac{1}{2}\sigma_x(1+\cos 2\theta) + \tfrac{1}{2}\sigma_y(1-\cos 2\theta)$$

i.e. $$\boldsymbol{\sigma_\theta = \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta} \qquad ...(5)$$

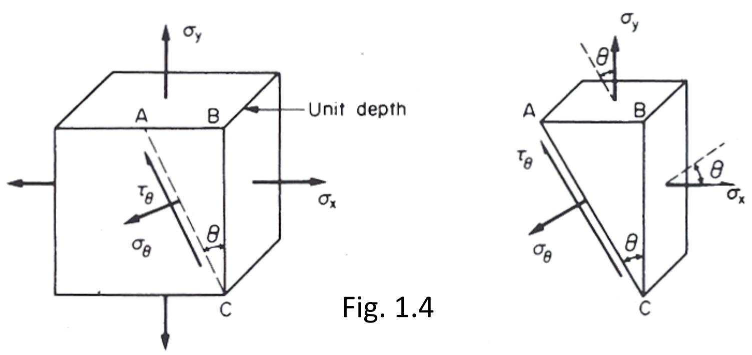



Fig. 1.4

Resolving parallel to AC:

$$\tau_\theta \times AC \times 1 = \sigma_x \times BC \times 1 \times \sin\theta - \sigma_y \times AB \times 1 \times \cos\theta$$

$$\tau_\theta = \sigma_x \cos\theta \sin\theta - \sigma_y \cos\theta \sin\theta$$

$$\therefore \quad \boldsymbol{\tau_\theta = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta} \quad \text{...(6)}$$

The maximum direct stress will equal be equal to $\sigma_x$ or $\sigma_y$, whichever is the greater, when $\theta$ = 0 or 90°. The maximum shear stress in the plane of the applied stresses occurs when $\theta$ = 45°,

i.e.

$$\boldsymbol{\tau_{max} = \tfrac{1}{2}(\sigma_x - \sigma_y)} \quad \text{...(7)}$$

## 1.4. <u>Material subjected to combined direct and shear stresses</u>

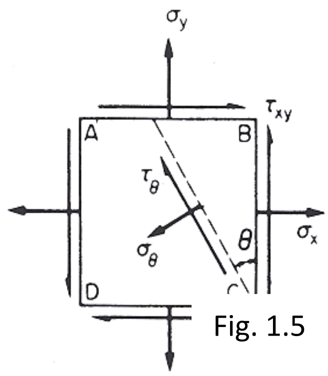



Fig. 1.5

Consider the complex stress system shown in Fig. 1.5 acting on an element of material. The stresses $\sigma_x$ and $\sigma_y$ may be compressive or tensile and may be the result of direct forces or bending. The shear stresses may be as shown or completely reversed and occur as a result of either shear forces or torsion.

The diagram thus represents a complete stress system for any condition of applied load in two dimensions and represents an addition of the stress systems previously considered in §1.2 and 1.3. The formulae obtained in these sections may therefore be combined to give

$$\boldsymbol{\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta}$$

and

$$\boldsymbol{\tau_\theta = \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta} \quad \text{...(9)}$$

The maximum and minimum stresses which occur on any plane in the material can now bedetermined as follows:

For $\sigma_\theta$ to be a maximum or minimum $\dfrac{d\sigma_\theta}{d\theta} = 0$

Now

$$\sigma_\theta = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\therefore \quad \frac{d\sigma_\theta}{d\theta} = -(\sigma_x - \sigma_y)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

or

$$\boldsymbol{\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}} \quad \text{...(10)}$$

$\therefore$ from Fig.1.6

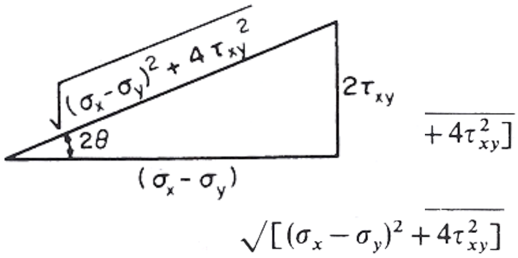



Fig. 1.6

Therefore substituting in eqn. (8), the maximum and minimum direct stresses are given by

$$\boldsymbol{\sigma_1} \quad \text{or} \quad \boldsymbol{\sigma_2} = \tfrac{1}{2}(\sigma_x+\sigma_y) + \tfrac{1}{2}\frac{(\sigma_x-\sigma_y)(\sigma_x-\sigma_y)}{\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]}} + \frac{\tau_{xy}\times 2\tau_{xy}}{\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]}}$$

$$= \tfrac{1}{2}(\boldsymbol{\sigma_x}+\boldsymbol{\sigma_y}) \pm \tfrac{1}{2}\sqrt{[(\boldsymbol{\sigma_x}-\boldsymbol{\sigma_y})^2+4\boldsymbol{\tau}_{xy}^2]} \qquad \text{...(11)}$$

These are then termed the ***principal stresses*** of the system.

The solution of eqn. (10) yields two values of 2θ separated by $180^0$, i.e. two values of θ separated by 90°. Thus the two principal stresses occur on mutually perpendicular planes termed ***principal planes***, and substitution for θ from eqn. (10) into the shear stress expression eqn. (9) will show that $\tau_\theta = 0$ on the principal planes.

The complex stress system of Fig. 1.5 can now be reduced to the equivalent system of principal stresses shown in Fig. 1.7.

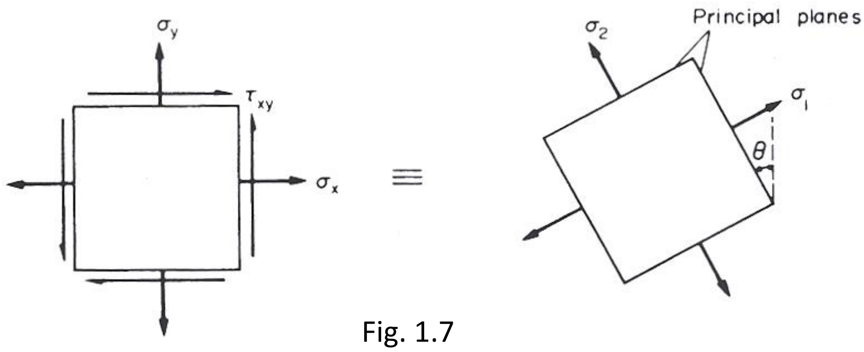



Fig. 1.7

From eqn. (7) the maximum shear stress present in the system is given by

...(12)

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2)$$

$$= \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \qquad ...(13)$$

and this occurs ***on planes at 45° to the principal planes***.

This result could have been obtained using a similar procedure to that used for determining the principal stresses, i.e. by differentiating expression (9), equating to zero and substituting the resulting expression for θ.

**1.5. <u>Principal plane inclination in terms of the associated principal stress</u>**

It has been stated in the previous section that expression (10), namely

$$\tan 2\theta = \frac{2\tau_{xy}}{(\sigma_x - \sigma_y)}$$

yields two values of θ, i.e. the inclination of the two principal planes on which the principal stresses $\sigma_1$ and $\sigma_2$ act. It is uncertain, however, which stress acts on which plane unless eqn. (8) is used, substituting one value of θ obtained from eqn. (10) and observing which one of the two principal stresses is obtained. The following alternative solution is therefore to be preferred.

Consider once again the equilibrium of a triangular block of material of unit depth (Fig. 1.8); This time AC is a principal plane on which a principal stress $\sigma_p$ acts, and the shearstress is zero (from the property of principal planes).

Resolving forces horizontally,

$$(\sigma_x \times BC \times 1) + (\tau_{xy} \times AB \times 1) = (\sigma_p \times AC \times 1)\cos\theta$$

$$\sigma_x + \tau_{xy}\tan\theta = \sigma_p$$

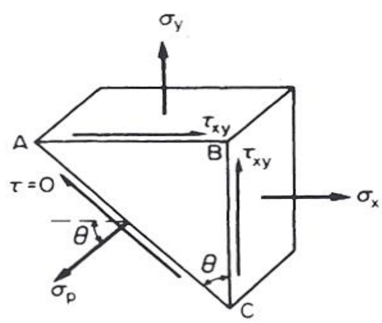


Fig. 1.8

$$\therefore \qquad \tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}}$$

Thus we have an equation for the inclination of the principal planes in terms of the principal stress. If, therefore, the principal stresses are determined and substituted in the above equation, each will give the corresponding angle of the plane on which it acts and there can then be no confusion.

The above formula has been derived with two tensile direct stresses and a shear stress system, as shown in the figure; Should any of these be reversed in action, then the appropriateminus sign must be inserted in the equation.

## 1.6. Graphical solution—Mohr's stress circle

Consider the complex stress system of Fig. 1.5 (Pg. 4). As stated previously this represents a complete stress system for any condition of applied load in two dimensions.

In order to find graphically the direct stress $\sigma_\theta$ and shear stress $\tau_\theta$ on any plane inclined at θ to the plane on which $\sigma_x$ acts, proceed as follows:

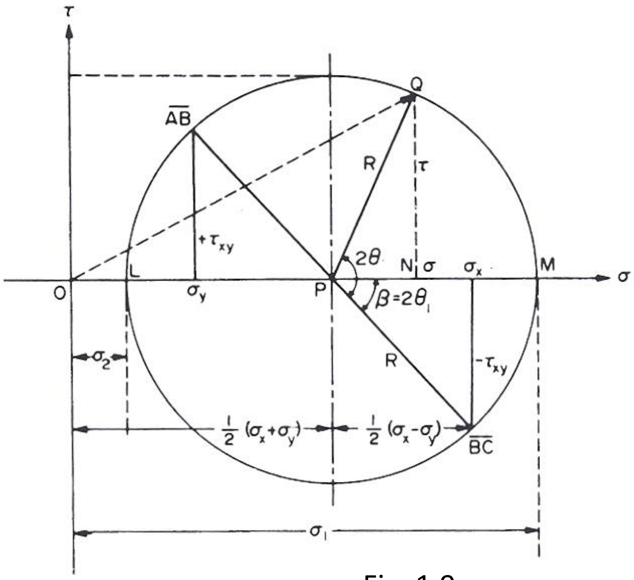


Fig. 1.9

(1) Label the block ABCD.

(2) Set up axes for direct stress (as abscissa) and shear stress (as ordinate) (Fig. 1.9).

(3) Plot the stresses acting on two adjacent faces, e.g. AB and BC, using the following sign conventions:

**Direct stresses:** tensile, positive; compressive, negative;

**shear stresses:** tending to turn block clockwise, positive; tending to turn block counterclockwise, negative.

This gives two points on the graph which may then be labelled $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

(4) Join $\overline{AB}$ and $\overline{BC}$.

(5) The point P where this line cuts the σ axis is then the centre of Mohr's circle, and the line is the diameter; therefore the circle can now be drawn.

***Proof***

Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θwith $\overline{BC}$, and drop a perpendicular from Q to meet the σ axis at N.

Coordinates of Q:

$$ON = OP + PN = \tfrac{1}{2}(\sigma_x + \sigma_y) + R\cos(2\theta - \beta)$$
$$= \tfrac{1}{2}(\sigma_x + \sigma_y) + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

But $$R\cos\beta = \tfrac{1}{2}(\sigma_x - \sigma_y) \quad \text{and} \quad R\sin\beta = \tau_{xy}$$

$$\therefore \quad ON = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$$

On inspection this is seen to be eqn. (8) for the direct stress $\sigma_\theta$on the plane inclined at θ to BC in Fig. 1.5.

Similarly, $$QN = R\sin(2\theta - \beta)$$
$$= R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$$
$$= \tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$$

Again, on inspection this is seen to be eqn. (9) for the shear stress $\tau_\theta$ on the plane inclined at θ to BC.

Thus the coordinates of Q are the normal and shear stresses on a plane inclined at θ to BC in the original stress system.

**N.B.**: Single angle $\overline{BC}$PQ is 2θ on Mohr's circle and not θ, it is evident that angles are doubled on Mohr's circle. This is the only difference, however, as they are measured in the same direction and from the same plane in both figures (in this case counterclockwise from $\overline{BC}$).

**Further points to note are:**

**(1)** The direct stress is a maximum when Q is at M, i.e. OM is the length representing the maximum principal stress $\sigma_1$ and $2\theta_1$ gives the angle of the plane $\theta_1$ from BC. Similarly, OL is the other principal stress.

**(2)** The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle. This follows since shear stresses and complementary shear stresses have the same value; therefore the centre of the circle will always lie on the σ axis midway between $\sigma_x$ and $\sigma_y$.

**(3)** From the above point the direct stress on the plane of maximum shear must be midway between $\sigma_x$ and $\sigma_y$, i.e. $\frac{1}{2}(\sigma_x + \sigma_y)$.

(4) The shear stress on the principal planes is zero.

(5) Since the resultant of two stresses at 90° can be found from the parallelogram of vectors as the diagonal, as shown in Fig. 1.10, the resultant stress on the plane at θ to BC is given by OQ on Mohr's circle.

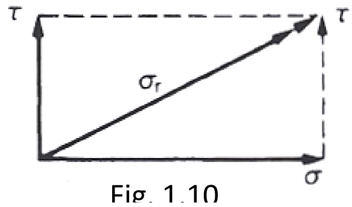


Fig. 1.10

The graphical method of solution of complex stress problems using Mohr's circle is a very powerful technique since all the information relating to any plane within the stressed element is contained in the single construction. It thus provides a convenient and rapid means of solution which is less prone to arithmetical errors and is highly recommended.

## Examples

**1)** A circular bar 40mm diameter carries an axial tensile load of 100 kN. What is the value of the shear stress on the planes on which the normal stress has a value of 50 MN/m$^2$ tensile?

## Solution

Tensile stress $$\sigma_y = \frac{F}{A} = \frac{100 \times 10^3}{\pi \times (0.02)^2} = 79.6\ \mathrm{MN/m^2}$$

Now the normal stress on an oblique plane is given by eqn. (1):

$$\sigma_\theta = \sigma_y \sin^2\theta$$
$$50 \times 10^6 = 79.6 \times 10^6 \sin^2\theta$$
$$\theta = 52^\circ\, 28'$$

The shear stress on the oblique plane is then given by eqn. (2):

$$\tau_\theta = \tfrac{1}{2}\sigma_y \sin 2\theta$$
$$= \tfrac{1}{2} \times 79.6 \times 10^6 \times \sin 104^\circ\, 56'$$
$$= 38.6 \times 10^6$$

The required shear stress is 38.6 MN/m$^2$.

**2)** Under certain loading conditions the stresses in the walls of a cylinder are as follows:

(a) 80 MN/m$^2$ tensile;

(b) 30 MN/m$^2$ tensile at right angles to (a);

(c) shear stresses of 60 MN/m$^2$ on the planes on which the stresses (a) and (b) act; the shear couple acting on planes carrying the 30 MN/m$^2$ stress is clockwise in effect.

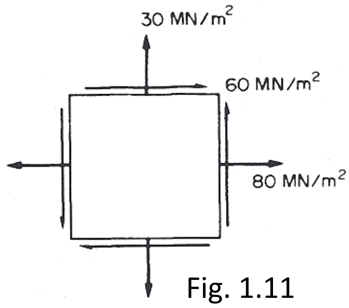


Fig. 1.11

Calculate the principal stresses and the planes on which they act. What would be the effect on these results if owing to a change of loading (a) becomes compressive while stresses (b) and (c) remain unchanged?

**Solution**

The principal stresses are given by the formula

$$\begin{aligned}
\sigma_1 \quad \text{and} \quad \sigma_2 &= \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]} \\
&= \tfrac{1}{2}(80 + 30) \pm \tfrac{1}{2}\sqrt{[(80 - 30)^2 + (4 \times 60^2)]} \\
&= 55 \pm 5\sqrt{(25 + 144)} \\
&= 55 \pm 65 \\
\therefore \qquad \sigma_1 &= \mathbf{120\ MN/m^2} \\
\text{and} \qquad \sigma_2 &= \mathbf{-10\ MN/m^2} \qquad \text{(i.e. compressive)}
\end{aligned}$$

The planes on which these stresses act can be determined from eqn. (15),

$$\begin{aligned}
\text{i.e.} \qquad \tan\theta_1 &= \frac{\sigma_p - \sigma_x}{\tau_{xy}} \\
\therefore \qquad \tan\theta_1 &= \frac{120 - 80}{60} = 0.6667 \\
\therefore \qquad \theta_1 &= \mathbf{33^\circ\ 41'} \\
\text{Also} \qquad \tan\theta_2 &= = \frac{-10 - 80}{60} = 1.50 \\
\therefore \qquad \theta_2 &= \mathbf{-56^\circ\ 19'} \quad \textbf{or} \quad \mathbf{123^\circ\ 41'}
\end{aligned}$$

**N.B.:**The resulting angles are at 90° to each other as expected.

If the loading is now changed so that the 80 MN/m$^2$ stress becomes compressive:

$$\sigma_1 = \tfrac{1}{2}(-80+30)+\tfrac{1}{2}\sqrt{[(-80-30)^2+(4\times 60^2)]}$$
$$= -25+5\sqrt{(121+144)}$$
$$= -25+81.5 = \mathbf{56.5\,MN/m^2}$$

and $\sigma_2 = -25-81.5 = \mathbf{-106.5\,MN/m^2}$

Then $\tan\theta_1 = \dfrac{56.5-(-80)}{60} = 2.28$

$\therefore$ $\theta_1 = \mathbf{66°\,19'}$

and $\theta_2 = 66°\,19' + 90 = \mathbf{156°\,19'}$

**<u>Mohr's circle solutions</u>**

In the first part of the question the stress system and associated Mohr's circle are as drawn in Fig. 1.12.

**By measurement:**

$\sigma_1 = \mathbf{120\,MN/m^2}$ **tensile**

$\sigma_2 = \mathbf{10\,MN/m^2}$ **compressive**

and

$\theta_1 = \mathbf{34°}$ **counterclockwise from** $\boldsymbol{BC}$

$\theta_2 = \mathbf{124°}$ **counterclockwise from** $\boldsymbol{BC}$

When the 80 MN/m$^2$ stress is reversed, the stress system is that in Fig. 1.13, giving Mohr's circle as drawn.

The required values are then:

$\sigma_1 = \mathbf{56.5\,MN/m^2}$ **tensile**

$\sigma_2 = \mathbf{106.5\,MN/m^2}$ **compressive**

and

$\theta_1 = \mathbf{66°\,15'}$ **counterclockwise to** $\boldsymbol{BC}$

$\theta_2 = \mathbf{156°\,15'}$ **counterclockwise to** $\boldsymbol{BC}$

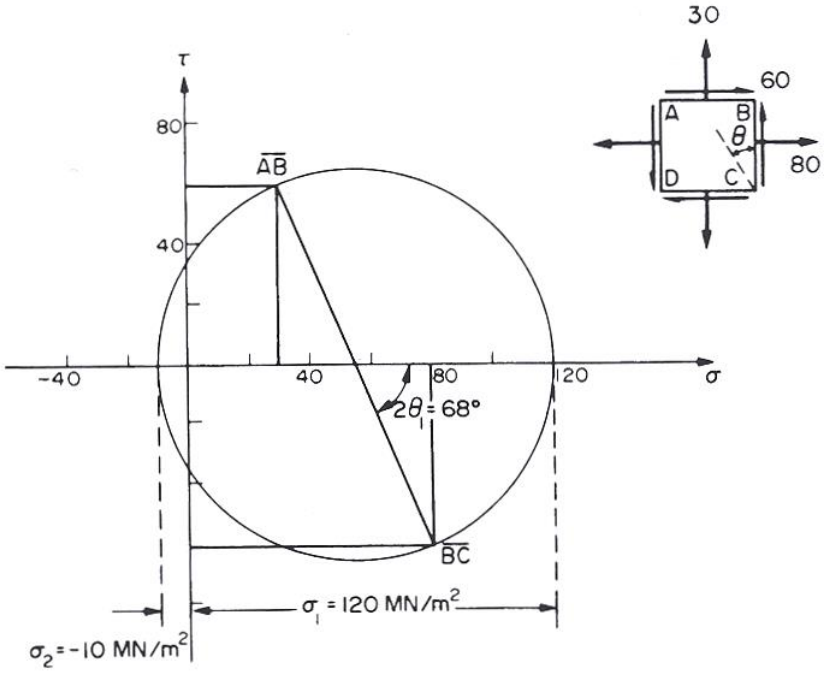



Fig. 1.12

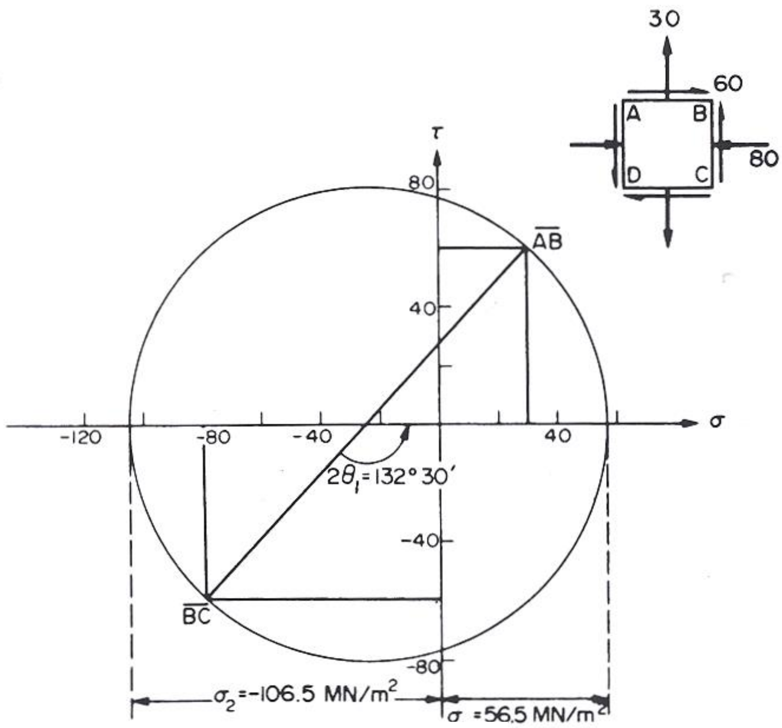



Fig. 1.13

**3)** A material is subjected to two mutually perpendicular direct stresses of 80 MN/m$^2$ tensile and 50 MN/m$^2$ compressive, together with a shear stress of 30 MN/m$^2$. The shear couple acting on planes carrying the 80 MN/m$^2$ stress is clockwise in effect. Calculate

(a) the magnitude and nature of the principal stresses;

(b) the magnitude of the maximum shear stresses in the plane of the given stress system;

(c) the direction of the planes on which these stresses act.

Confirm your answer by means of a Mohr's stress circle diagram, and from the diagram determine the magnitude of the normal stress on a plane inclined at 20° counterclockwise to the plane on which the 50 MN/m$^2$ stress acts.

**Solution**

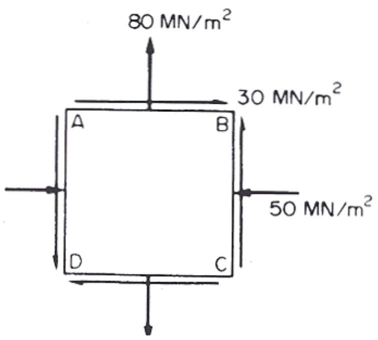


Fig. 1.14

(a) To find the principal stresses:

$$\sigma_1 \text{ and } \sigma_2 = \tfrac{1}{2}(\sigma_x+\sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]}$$
$$= \tfrac{1}{2}(-50+80) \pm \tfrac{1}{2}\sqrt{[(-50-80)^2+(4\times 900)]}$$
$$= 5[3 \pm \sqrt{(169+36)}] = 5[3 \pm 14.31]$$
$$\therefore \quad \sigma_1 = 86.55 \text{ MN/m}^2$$
$$\sigma_2 = -56.55 \text{ MN/m}^2$$

The principal stresses are

86.55 MN/m$^2$ tensile and 56.55 MN/rn$^2$ compressive

(b) To find the maximum shear stress:

$$\tau_{max} = \frac{\sigma_1-\sigma_2}{2} = \frac{86.55-(-56.55)}{2} = \frac{143.1}{2} = 71.6 \text{ MN/m}^2$$

Maximum shear stress = **71.6 MN/m²**

(c) To find the directions of the principal planes:

$$\tan\theta_1 = \frac{\sigma_p-\sigma_x}{\tau_{xy}} = \frac{86.55-(-50)}{30}$$
$$= \frac{136.55}{30} = 4.552$$

$$\therefore \quad \theta_1 = \mathbf{77^\circ\,36'}$$
$$\therefore \quad \theta_2 = 77^\circ\,36' + 90^\circ = \mathbf{167^\circ\,36'}$$

The principal planes are inclined at 77° 36’ to the plane on which the 50 MN/m$^2$ stress acts. The maximum shear planes are at 45° to the principal planes.

**Mohr’s circle solution**

The stress system shown in Fig. 1.14 gives the Mohr’s circle in Fig. 1.15.

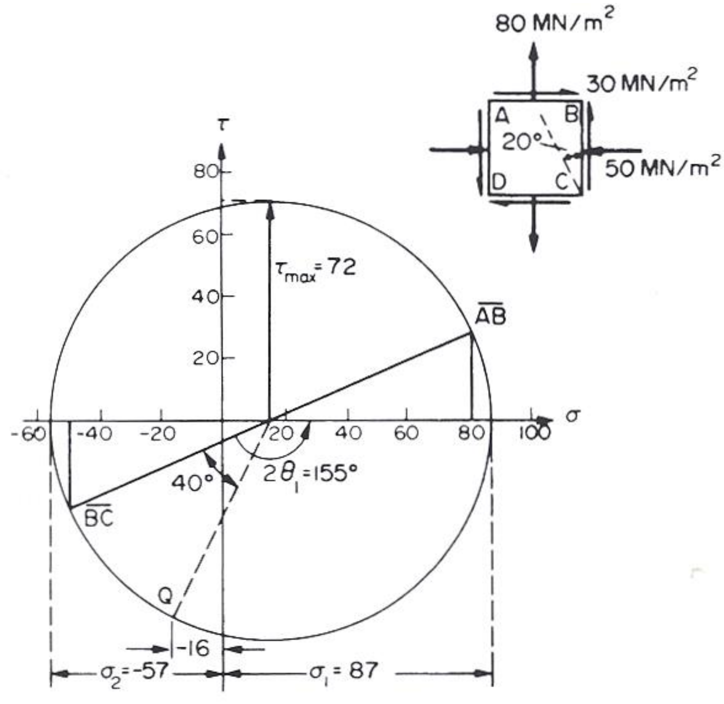


Fig. 1.15

By measurement

$$\sigma_1 = 87\ \mathbf{MN/m^2}\ \textbf{tensile}$$

$$\sigma_2 = 57\ \mathbf{MN/m^2}\ \textbf{compressive}$$

$$\tau_{max} = 72\ \mathbf{MN/m^2}$$

and $$\theta_1 = \frac{155^\circ}{2} = 77^\circ\ 30'$$

The direct or normal stress on a plane inclined at 20° counterclockwise to BC is obtained by measuring from BC on the Mohr’s circle through 2 x 20° = 40° in the same direction.

This gives σ = 16 MN/m$^2$ compressive

**4)** At a given section a shaft is subjected to a bending stress of 20 MN/m$^2$ and a shear stress of 40 MN/m$^2$. Determine:

(a) the principal stresses;

(b) the directions of the principal planes;

(c) the maximum shear stress and the planes on which this acts;

(d) the tensile stress which, acting alone, would produce the same maximum shear stress;

(e) the shear stress which, acting alone, would produce the same maximum tensile principal stress.

## Solution

(a) The bending stress is a direct stress and can be treated as acting on the x axis, so that = 20 MN/m$^2$ since no other direct stresses are given, $\sigma_y$= 0.

Principal stress

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$
$$= \tfrac{1}{2} \times 20 + \tfrac{1}{2}\sqrt{[20^2 + (4 \times 40^2)]}$$
$$= 10 + 5\sqrt{(68)} = 10 + 5 \times 8.246$$
$$= \mathbf{51.23\ MN/m^2}$$

and $\sigma_2 = 10 - 41.23$

$$= \mathbf{-31.23\ MN/m^2}$$

(b) Then $\tan\theta_1 = \dfrac{\sigma_p - \sigma_x}{\tau_{xy}} = \dfrac{51.23 - 20}{40} = \dfrac{31.23}{40} = 0.7808$

$\therefore$ $\theta_1 = \mathbf{37^\circ\ 59'}$

and $\tan\theta_2 = \dfrac{-31.23 - 20}{40} = \dfrac{-51.23}{40} = -1.2808$ both angles being measured

$\therefore$ $\theta_2 = \mathbf{-52^\circ\ 1'}$ **or** $\mathbf{127^\circ\ 59'}$

counterclockwise from the plane on which the 20 MN/m$^2$ stress acts.

(c) Maximum shear stress

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{51.23 - (-31.23)}{2}$$
$$= \frac{82.46}{2} = \mathbf{41.23\ MN/m^2}$$

This acts on planes at 45° to the principal planes,

i.e. **at 82° 59′ or −7° 1′**

(d) Maximum shear stress $\tau_{max} = \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$

Thus if a tensile stress is to act alone to give the same maximum shear stress ($\sigma_y$ = 0 and $\tau_{xy}$ = 0):

$$\text{maximum shear stress} = \tfrac{1}{2}\sqrt{(\sigma_x^2)} = \tfrac{1}{2}\sigma_x$$
$$41.23 = \tfrac{1}{2}\sigma_x$$
i.e. $$\sigma_x = \mathbf{82.46\ MN/m^2}$$

The required tensile stress is 82.46 MN/m$^2$.

(e) Principal stress

$$\sigma_1 = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

Thus if a shear stress is to act alone to give the same principal stress ($\sigma_x$= $\sigma_y$= 0):

$$\sigma_1 = \tfrac{1}{2}\sqrt{(4\tau_{xy}^2)} = \tau_{xy}$$
$$51.23 = \tau_{xy}$$

The required shear stress is 51.23 MN/m$^2$.

**<u>Mohr's circle solutions</u>**

(a), (b), (c) The stress system and corresponding Mohr's circle are as shown in Fig. 1.16.

By measurement:

(a) $\sigma_1 \simeq \mathbf{51\ MN/m^2\ tensile}$

$\sigma_2 \simeq \mathbf{31\ MN/m^2\ compressive}$

(b) $\theta_1 = \dfrac{76^\circ}{2} = \mathbf{38^\circ}$

$\theta_2 = 38^\circ + 90^\circ = \mathbf{128^\circ}$

(c) $\tau_{max} \simeq \mathbf{41\ MN/m^2}$

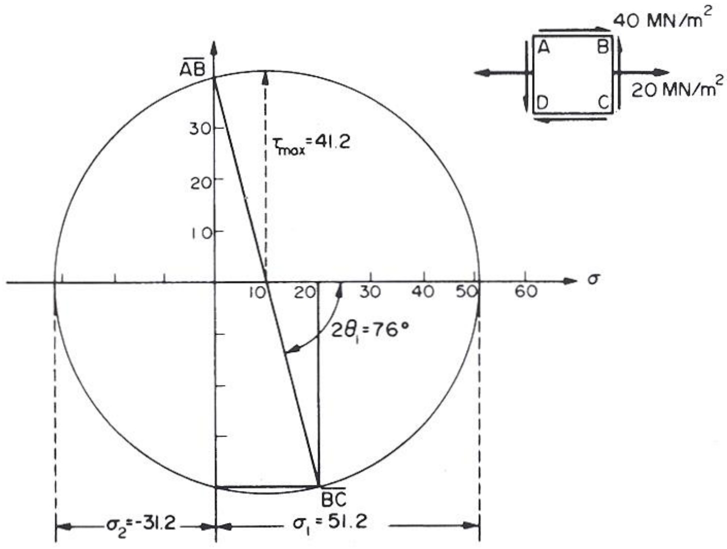



Fig. 1.16

Angle of maximum shear plane

$$= \frac{166}{2} = \mathbf{83°}$$

(d) If a tensile stress $\sigma_x$, is to act alone to give the same maximum shear stress, then

$\sigma_y = 0$, $\tau_{xy} = 0$ and $\tau_{max} = 41$ MN/m$^2$. The Mohr's circle therefore has a radius of 41 MN/m$^2$ andpasses through the origin (Fig. 1.17).

Hence the required tensile stress is 82 MN/m$^2$.

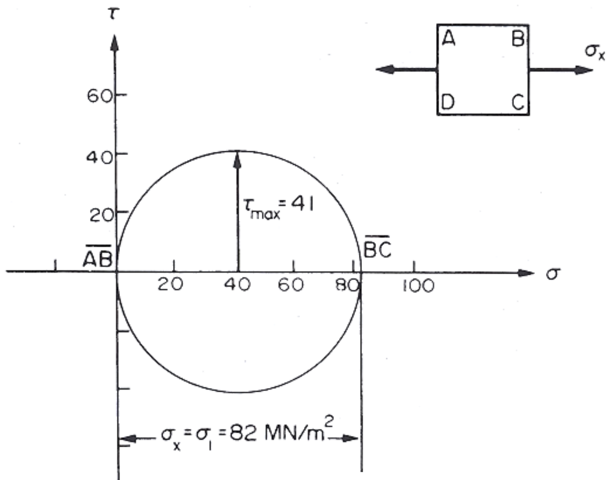


Fig. 1.17

(e) If a shear stress is to act alone to produce the same principal stress, $\sigma_x$, = 0, $\sigma_y = 0$ and $\sigma_1 = 51$ MN/m$^2$. The Mohr's circle thus has its centre at the origin and passes through $\sigma = 51$ MN/m$^2$ (Fig. 1.18).

Hence the required shear stress is 51 MN/m$^2$.

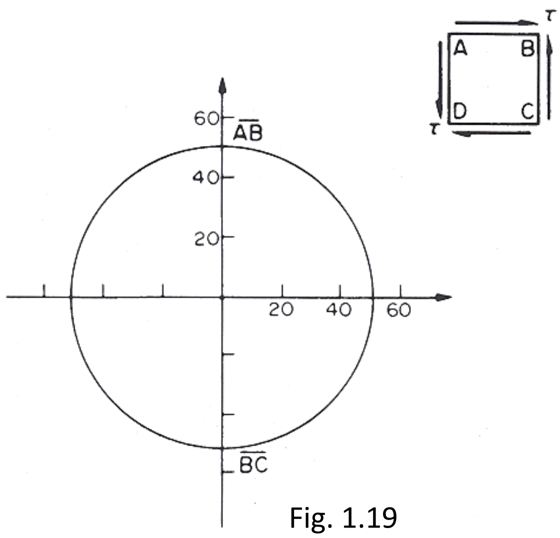


Fig. 1.19

**5)** At a point in a piece of elastic material direct stresses of 90 MN/m$^2$ tensile and 50 MN/m$^2$ compressive are applied on mutually perpendicular planes. The planes are also subjected to a shear stress. If the greater principal stress is limited to 100 MN/m$^2$ tensile, determine:

(a) the value of the shear stress;

(b) the other principal stress;

(c) the normal stress on the plane of maximum shear;

(d) the maximum shear stress.

Make a neat sketch showing clearly the positions of the principal planes and planes of maximum shear stress with respect to the planes of the applied stresses.

**Solution**

(a) Principal stress $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$

This is limited to 100 MN/m$^2$ therefore shear stress $\tau_{xy}$, is given by

$$100 = \tfrac{1}{2}(90 - 50) + \tfrac{1}{2}\sqrt{[(90 + 50)^2 + 4\tau_{xy}^2]}$$

$$\therefore \quad 200 = 40 + 10\sqrt{[14^2 + 0.04\tau_{xy}^2]}$$

$$\therefore \quad \tau_{xy} = \sqrt{\left(\frac{16^2 - 14^2}{0.04}\right)} = \sqrt{\left(\frac{256 - 196}{0.04}\right)} = \frac{\sqrt{60}}{0.2}$$

$$= \mathbf{38.8\ MN/m^2}$$

The required shear stress is 38.8 MN/m$^2$.

(b) The other principal stress $\sigma_2$ is given by

$$\sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2]}$$

$$= \tfrac{1}{2}[(90 - 50) - 10\sqrt{(14^2 + 60)}] = \frac{40 - 10\sqrt{(256)}}{2}$$

$$= \frac{40 - 160}{2} \doteq -\mathbf{60\ MN/m^2}$$

The other principal stress is 60 MN/m$^2$ compressive.

(c) The normal stress on the plane of maximum shear

$$= \frac{\sigma_1 + \sigma_2}{2} = \frac{100 - 60}{2}$$

$$= \mathbf{20\ MN/m^2}$$

The required normal stress is 20 MN/m$^2$ tensile.

(d) The maximum shear stress is given by

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100 + 60}{2}$$

$$= \mathbf{80\ MN/m^2}$$

The maximum shear stress is 80 MN/m$^2$.

In order tobe able to draw the required sketch (Fig. 1.20) to indicate the relative positions of the planes on which the above stresses act, the angles of the principal planes are required. These are given by

$$\tan\theta = \frac{\sigma_p - \sigma_x}{\tau_{xy}} = \frac{100-(-50)}{38.8}$$

$$= \frac{150}{38.8} = 3.87$$

$$\therefore \qquad \theta_1 = 75^\circ\,30'$$

To the plane on which the 50 MN/m$^2$ stress acts.

The required sketch is then shown in Fig. 1.20.

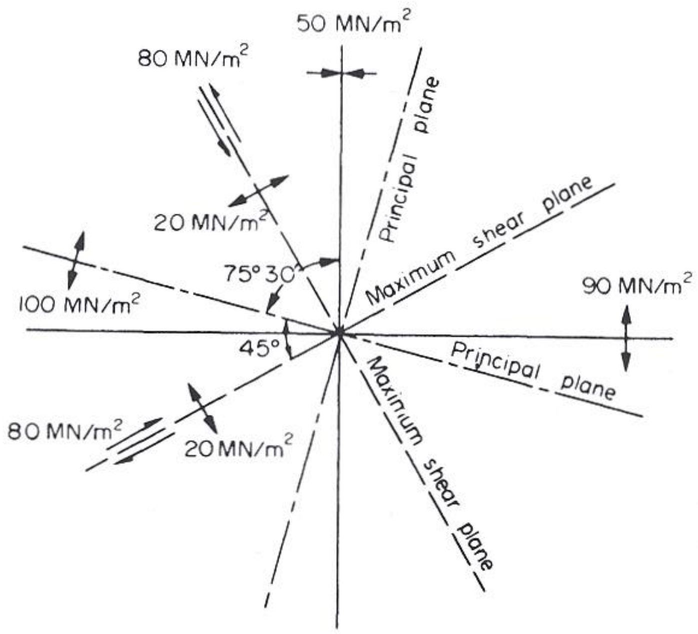



Fig. 1.20

**Mohr's circle solution**

The stress system is as shown in Fig. 1.20. The centre of the Mohr's circle is positioned midway between the two direct stresses given, and the radius is such that $\sigma_1$ = 100 MN/m$^2$.

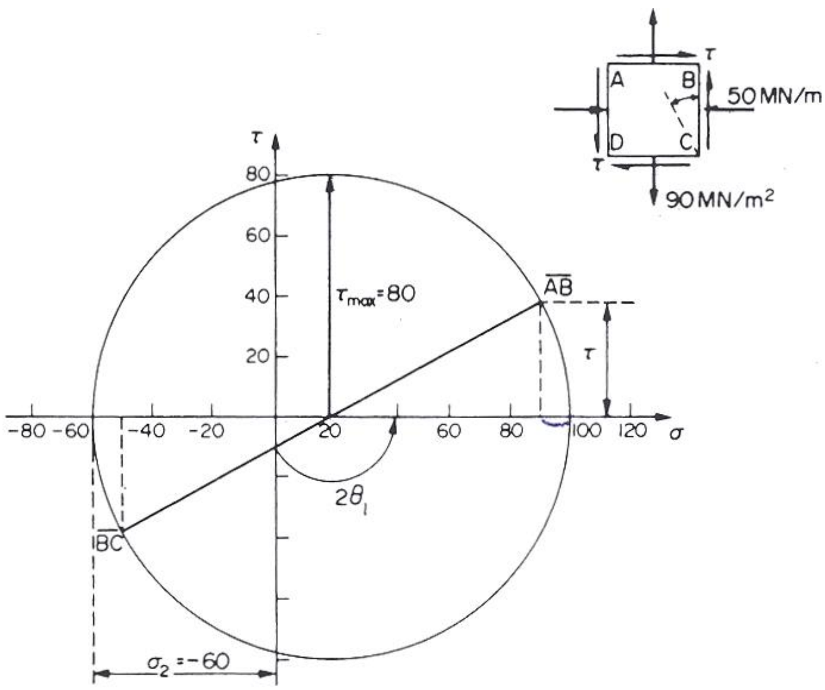
By measurement:

$$\tau = 39\,\mathrm{MN/m^2}$$

$$\sigma_2 = 60\,\mathrm{MN/m^2}\ \text{compressive}$$

$$\tau_{max} = 80\,\mathrm{MN/m^2}$$

$$\theta_1 = \frac{151}{2} = 75^\circ\,30' \text{ to } BC\text{, the plane on which the } 50\,\mathrm{MN/m^2} \text{ stress acts}$$



τ
A
B
50 MN/m
D
C
τ
90 MN/m²
τ
80
60
40
20
τmax = 80
AB
τ
-80
-60
-40
-20
20
60
80
100
120
σ
2θ1
BC
σ2 = -60